Some differentials on colored Khovanov-Rozansky link homology

Paul Wedrich
Imperial College London

MATRIX, 14th December 2016

This work was supported by an EPSRC doctoral training grant at the University of Cambridge, the Leverhulme Research Grant RP2013-K-017 to Dorothy Buck, and the Max Planck Institute for Mathematics.

Plan

1. Motivation
2. sl(N) link homologies
3. Deformations
4. Physical structure, HOMFLY-PT homology

A zoo of link polynomials

Fact

The Jones polynomial is uniquely determined by its value on the unknot and the oriented skein relation:

$$q^2 V(\overcrossing) - q^{-2} V(\undercrossing) = (q - q^{-1}) V(\smoothing)$$

Varying this skein relation, we get other link polynomials:

- $q^N P_N(\overcrossing) - q^{-N} P_N(\undercrossing) = (q - q^{-1}) P_N(\smoothing)$ $\mathfrak{sl}_N$ polynomial.
- $a P_\infty(\overcrossing) - a^{-1} P_\infty(\undercrossing) = (q - q^{-1}) P_\infty(\smoothing)$ HOMFLY-PT polynomial
- $\Delta(\overcrossing) - \Delta(\undercrossing) = (q - q^{-1}) \Delta(\smoothing)$ Alexander-Conway polynomial

For framed links, you get even more invariants from cabling operations.

Reshetikhin-Turaev link invariants

The Reshetikhin-Turaev invariants for links in $\mathbb{R}^3$ give a function:

$$\{\text{triples } (L, \mathfrak{g}, \mathrm{col})\} \xrightarrow{\mathrm{RT}} \mathbb{Z}[q^{\pm 1}]$$

- L is a framed, oriented link in $\mathbb{R}^3$,
- $\mathfrak{g}$ is a complex semi-simple Lie algebra,
- $\mathrm{col}\colon \pi_0(L) \to \mathrm{Irrep}^{f.d.}(\mathfrak{g})$ is a coloring of the link components by finite-dimensional irreducible representations of $\mathfrak{g}$.

E.g. $V(L) = \mathrm{RT}(L, \mathfrak{sl}_2, \mathbb{C}^2)$ and $P_N(L) = \mathrm{RT}(L, \mathfrak{sl}_N, \mathbb{C}^N)$.

Question

How does this function depend on the three arguments?

For this talk:

- Lie algebras are of type A: $\mathfrak{g} = \mathfrak{sl}_N$ for various $N \in \mathbb{N}$.
- Mostly colorings by irreps $\mathbb{C}^N$ and $\bigwedge^k \mathbb{C}^N$ for $0 \leq k \leq N$.

Varying the coloring

The finite-dimensional irreducible representations of $\mathfrak{sl}_2$ are indexed by $k \in \mathbb{N}$ (in fact $V_k := \mathrm{Sym}^k(\mathbb{C}^2)$). Redundancies in this countably-infinite list of invariants?

Theorem (Garoufalidis-Lê)

Let K be a framed knot in $\mathbb{R}^3$. The sequence of colored Jones polynomials $(\mathrm{RT}(K, \mathfrak{sl}_2, \mathrm{Sym}^k(\mathbb{C}^2)))_{k \in \mathbb{N}}$ is q-holonomic.

So the sequence is governed by a linear recurrence relation (with coefficients polynomials in q and q^k) and, thus, determined by a finite part.

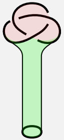
Analogous results hold for $\mathfrak{sl}_N$, for colored HOMFLY-PT polynomials, for links, with other sequences of colors... Garoufalidis-Lauda-Lê.

Varying the link?

Lie algebras and colorings can be varied in families. Some links come in families too, but let's take a different perspective.
Instead of just links, consider link embeddings in $\mathbb{R}^3$ and smooth cobordisms between them (in $\mathbb{R}^3 \times I$). Need categorified RT invariants:

$$\begin{array}{ccc} & \text{link homology LH} \dashrightarrow & \left\{ \begin{array}{c}\text{bi-graded vector spaces} \\ H_*(\text{graded chain complex})\end{array} \right\} \\ & & \downarrow \chi_q \\ \{\mathfrak{g}\text{-colored oriented links}\} & \xrightarrow{\text{RT}} & \mathbb{Z}[q^{\pm 1}] \end{array}$$

Cobordisms:



$$LH(0_1) \longrightarrow LH(3_1)$$

Ideally functorial under link cobordisms.

Goal for this talk

Overview about the rank- and color-dependence of type A link homologies.

Plan

1. Motivation
2. sl(N) link homologies
3. Deformations
4. Physical structure, HOMFLY-PT homology

Khovanov homology and its cousins

- 1999: Khovanov homology categorifies the Jones polynomial.

$$\mathrm{Kh}(\bigcirc) \cong H^*(\mathbb{CP}^1)\{-1\}$$

- 2004: Khovanov-Rozansky homology categorifies $\mathrm{RT}(-, \mathfrak{sl}_N, \mathbb{C}^N)$.

$$\mathrm{KhR}^N(\bigcirc) \cong H^*(\mathbb{CP}^{N-1})\{1-N\}$$

- 2009: Wu and Yonezawa extended Khovanov-Rozansky homology to a categorification of $\mathrm{RT}(-, \mathfrak{sl}_N, \bigwedge^k \mathbb{C}^N)$: colored $\mathfrak{sl}_N$ link homology.

$$\mathrm{KhR}^N(\bigcirc^k) \cong H^*(\mathrm{Gr}(k,N))\{k(k-N)\}$$

$$\mathrm{KhR}^N(K^1) = \mathrm{KhR}^N(K)$$

Flavors of colored $\mathfrak{sl}_N$ link homologies

1. Vanilla: via matrix factorizations, Khovanov-Rozansky, Wu, Yonezawa.
2. Representation theoretic: via category $\mathcal{O}$, Mazorchuk-Stroppel, Sussan.
3. Combinatorial: via cobordism or foam categories, Bar-Natan, Khovanov, Mackaay-Stošić-Vaz, Lauda-Queffelec-Rose.
4. Algebro-geometric: via affine Grassmannians, Cautis-Kamnitzer-Licata
5. Diagram-algebraic: via categorified tensor products, Webster.
6. Symplectic: via Floer homology, Seidel-Smith, Manolescu, Abouzaid.
7. Physical: via BPS state counting, Gukov-Schwarz-Vafa, et.al.

Two questions about the $\mathfrak{sl}_N$ link homology family

- What kind of geometric and topological information is accessible to it?
- What relations exist between its members?

Geometric and topological information

- Concordance homomorphisms, slice genus bounds, Rasmussen, Lobb, Wu.
- Thurston-Bennequin number bounds, Shumakovitch, Plamenevskaya, Ng.
- Splitting number bounds, Batson-Seed.
- Unknot detection, Kronheimer-Mrowka.
- ...

Fact

These results rely on spectral sequences between different link homologies.

Relations via deformation spectral sequences

- 2002: Lee constructed spectral sequences

$$\mathrm{Kh}(K) \rightsquigarrow \mathbb{C}^2 \qquad \mathrm{Kh}(L) \rightsquigarrow \mathbb{C}^{2^{|\pi_0(L)|}}$$

  leading to Rasmussen's concordance homomorphism.

- 2004: Gornik constructed spectral sequences

$$\mathrm{KhR}^N(K) \rightsquigarrow \mathbb{C}^N \qquad \mathrm{KhR}^N(L) \rightsquigarrow \mathbb{C}^{N^{|\pi_0(L)|}}$$

  leading to Lobb's concordance homomorphism.

- 2006: Mackaay-Vaz constructed spectral sequences:

$$\mathrm{KhR}^3(K) \rightsquigarrow \mathrm{KhR}^2(K) \oplus \mathbb{C}$$

More deformations

Theorem (folklore)

Let K be a knot and $\sum N_j = N$ with $N_j \in \mathbb{N}$, then there exists a deformation spectral sequence:

$$\mathrm{KhR}^N(K) \rightsquigarrow \bigoplus_j \mathrm{KhR}^{N_j}(K)$$

Theorem (Rose-W. 2015)

Let K be a knot and $\sum N_j = N$ with $N_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a deformation spectral sequence:

$$\mathrm{KhR}^N(K^k) \rightsquigarrow \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}^{N_j}(K^{k_j})$$

Mutatis mutandis for links.

Plan

1 Motivation

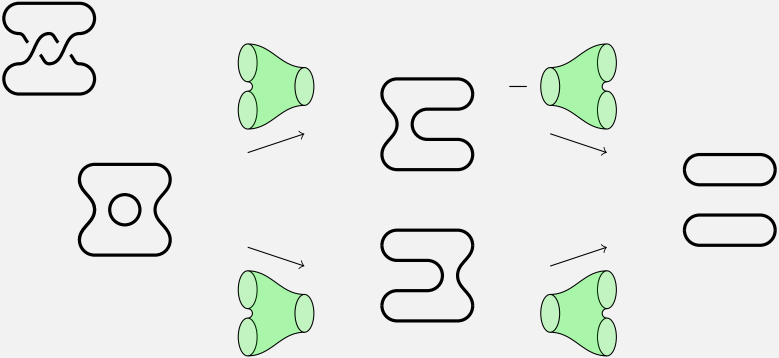
2 sl(N) link homologies

3 Deformations
- of Khovanov homology
- of $sl(N)$ link homologies

4 Physical structure, HOMFLY-PT homology

Bar-Natan's construction of Khovanov homology

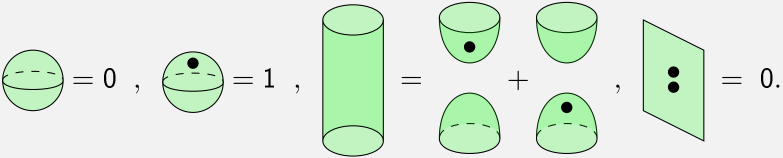
The cube of resolutions as a chain complex:



Bar-Natan's construction of Khovanov homology

Bar-Natan: Let Cob be the category consisting of

- Objects: formal direct sums of planar compact 1-manifolds,
- Morphisms: matrices of $\mathbb{C}$-linear combinations of "dotted" oriented cobordisms between 1-manifolds, modulo isotopy and local relations:

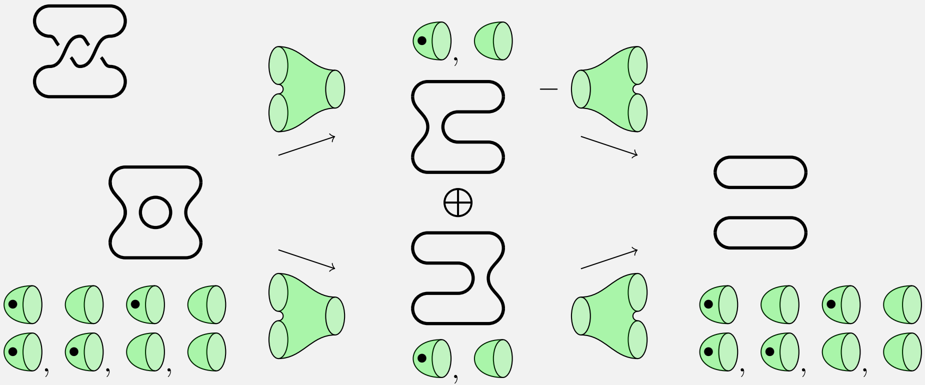


Cob admits a grading and $\mathrm{Hom}_{\mathrm{Cob}}(\emptyset, -)$ is a functor from Cob to graded vector spaces.

E.g. $\mathrm{Hom}_{\mathrm{Cob}}\left(\emptyset, \bigcirc\right) \cong \mathbb{C}\langle$  $\rangle \xrightarrow{\chi_q} q + q^{-1}$

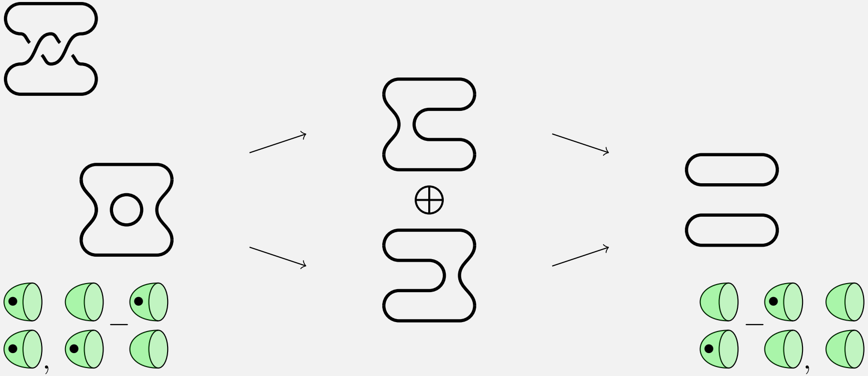
Bar-Natan's construction of Khovanov homology

After applying the TQFT:



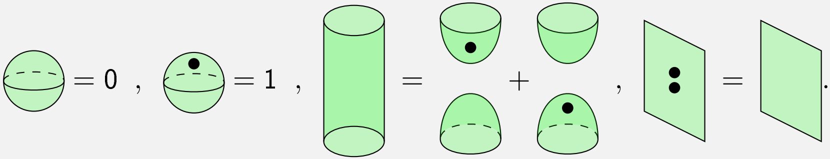
Bar-Natan's construction of Khovanov homology

After taking homology...



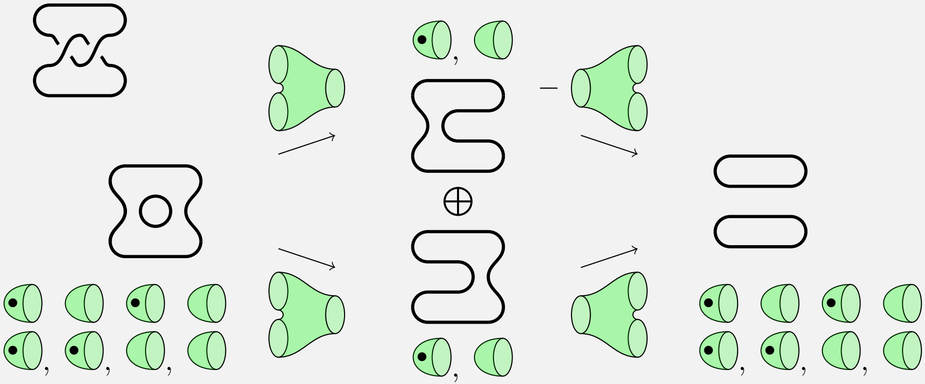
Lee's deformation of Khovanov homology

Bar-Natan, Morrison: Let Cob' be defined as before, but with the following set of relations



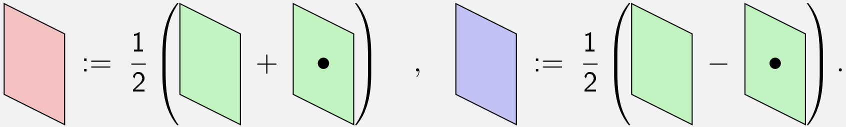
The cube of resolutions chain complex in Cob' is also a link invariant up to homotopy. Applying the functor $\mathrm{Hom}_{\mathrm{Cob}}(\emptyset, -)$ gives a complex of vector spaces, taking homology recovers Lee's deformation of Khovanov homology.

The cube of resolutions again...

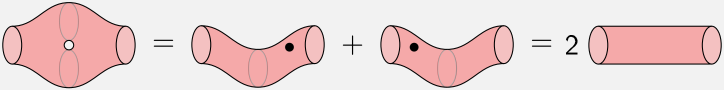


Lee's deformation of Khovanov homology

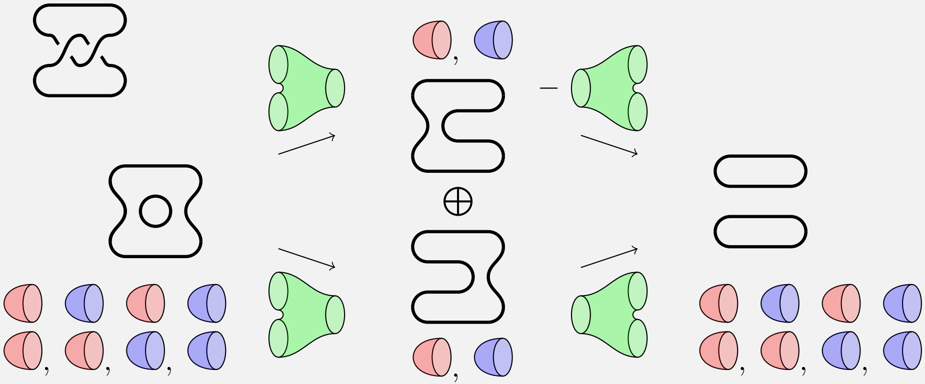
Have orthogonal idempotents:



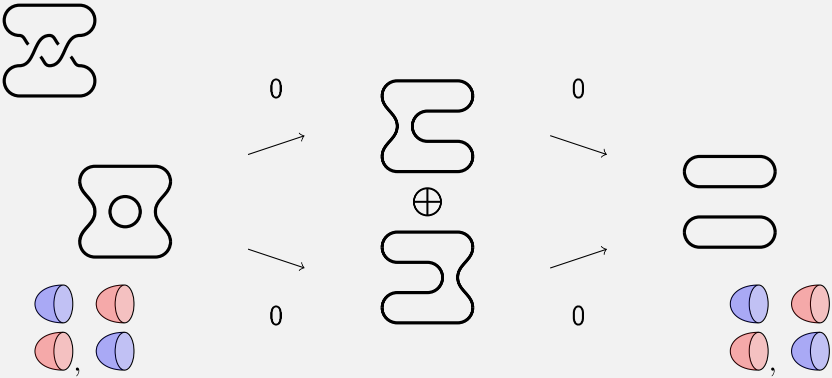
Can split every connected component of a cobordism into red and blue.
Red and blue pairs of pants are isomorphisms, e.g.



… after a change of basis



... and after Gaussian elimination



Proof strategy

Theorem (Rose-W. 2015)

Let K be a knot and $\sum N_j = N$ with $N_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a deformation spectral sequence:

$$\mathrm{KhR}^N(K^k) \rightsquigarrow \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}^{N_j}(K^{k_j})$$

1. Wu's spectral sequence
2. Unknot case
3. $\bigoplus$ decomposition
4. $\bigotimes$ decomposition
5. Identifying tensor factors

Proof Step 1 – Wu's spectral sequence

1. Wu's construction of colored $\mathfrak{sl}_N$ homology uses matrix factorization with potential X^N.

   Following ideas of Gornik and Rasmussen:
   Potential $P(X) = \prod_{\lambda \in \Sigma}(X - \lambda) \in \mathbb{C}[X]$ of degree N with root multiset Σ gives a singly-graded, filtered link homology theory $\mathrm{KhR}^{\Sigma}(-)$ and spectral sequences

$$\mathrm{KhR}^{N}(K^{k}) \rightsquigarrow \mathrm{KhR}^{\Sigma}(K^{k})$$

   It remains to compute $\mathrm{KhR}^{\Sigma}(K^{k})$ in terms of undeformed homologies.

Proof Step 2 – The unknot case

2. The link homology theory $\mathrm{KhR}^{\Sigma}(-)$ contains – and is controlled by – a $(1+1)$-dimensional TQFT. The corresponding commutative Frobenius algebra appears as the unknot invariant.
Let $\Sigma = \{\lambda_1^{N_1}, \ldots, \lambda_l^{N_l}\}$, $P(X) = \prod_j (X - \lambda_j)^{N_j}$, then we have:

$$\mathrm{KhR}^{\Sigma}(\bigcirc^1) \cong \frac{\mathbb{C}[X]}{\langle P(X) \rangle} \cong \bigoplus_j \frac{\mathbb{C}[X]}{\langle (X - \lambda_j)^{N_j} \rangle} \cong \bigoplus_j \mathrm{KhR}^{N_j}(\bigcirc^1).$$

Summands are indexed by roots of $P(X)$. And in the colored case:

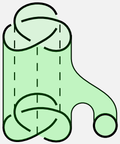
$$\mathrm{KhR}^{\Sigma}(\bigcirc^k) \cong \frac{\mathrm{Sym}[\mathbb{X}]}{\langle h_{N-k+i}(\mathbb{X} - \Sigma) \mid i > 1 \rangle} \cong \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}^{N_j}(\bigcirc^{k_j}).$$

Summands are indexed by size k multisubsets $\{\lambda_1^{k_1}, \ldots, \lambda_l^{k_l}\}$ of roots.

Proof Step 3 – The $\bigoplus$ decomposition

3. $\mathrm{KhR}^{\Sigma}(K^k)$ is a $\mathrm{KhR}^{\Sigma}(\bigcirc^k)$-module.

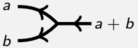
   If you believe in functoriality:

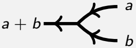
   

   If not, let's talk about foams . . .

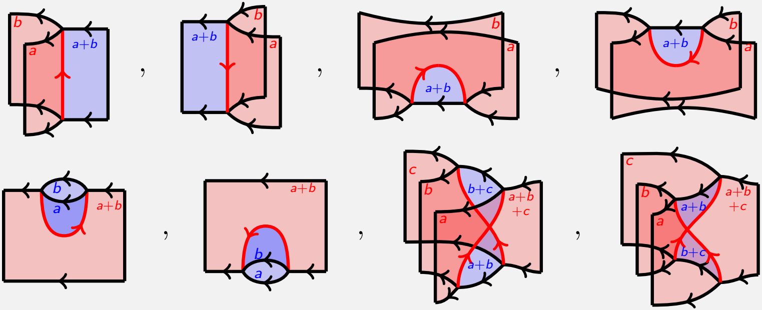
Foam technology

Lauda-Queffelec-Rose: The foam category N**Foam** consists of

- Objects: formal direct sums of leftward oriented, labeled, planar, trivalent graphs, built from: 

, 

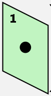
- Morphisms: matrices with entries being $\mathbb{C}$-linear combinations of decorated, singular cobordisms between webs generated by

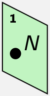


modulo isotopy and local relations.

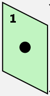
Foam technology

Lauda-Queffelec-Rose:

N**Foam**$^\bullet$: additional relation $\Big($ 

$\Big)^N =$ 

$= 0$

N**Foam**$^\Sigma$: additional relation $P\Big($ 

$\Big) = 0$

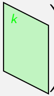
Colored $\mathfrak{sl}_N$ link homologies $\mathrm{KhR}^N(-)$ and their deformations $\mathrm{KhR}^\Sigma(-)$ can be computed via complexes in N**Foam**$^\bullet$ and N**Foam**$^\Sigma$:

- Link diagram + crossing replacement rule $\to$ cube of resolutions chain complex.
- Applying a representable functor gives a complex of vector spaces.
- Its homology is the desired link invariant.

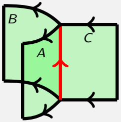
Proof Step 3 – The $\bigoplus$ decomposition

3. $\mathrm{KhR}^{\Sigma}(K^k)$ is a $\mathrm{KhR}^{\Sigma}(\bigcirc^k)$-module: In N**Foam**$^{\Sigma}$ we have

$$\mathrm{Decorations}\left(\text{[facet]} \right) \cong \mathrm{KhR}^{\Sigma}(\bigcirc^k).$$

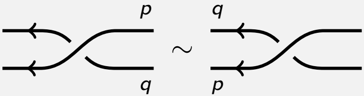


Facets split into sum over idempotent decorations $\leftrightarrow$ multisets $A \subset \Sigma$.

Compatibility: 

$= 0$ if $A \uplus B \neq C$

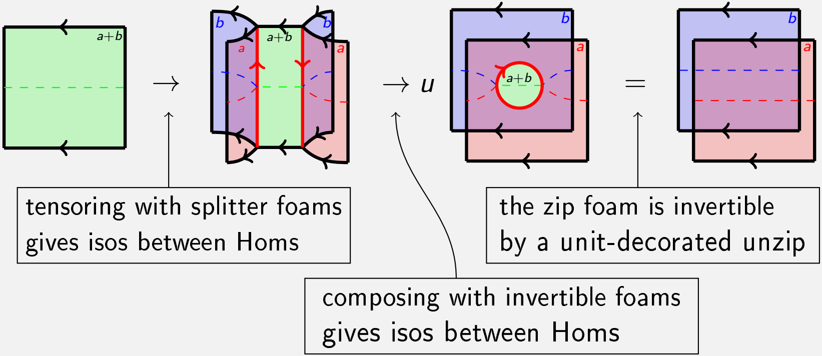
The actions on facets are compatible along link components:



For a knot we project on direct summand by choosing one idempotent $A = \{\lambda_1^{k_1}, \ldots, \lambda_l^{k_l}\}$, which propagates across crossings.

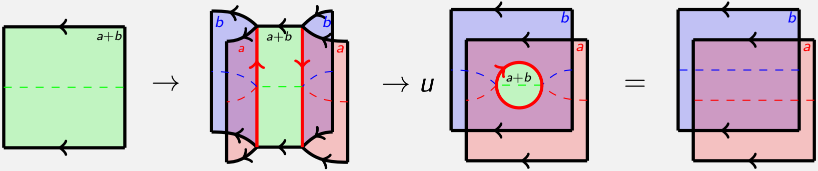
Proof Step 4 – The $\bigotimes$ decomposition

4. Look at summand of $\mathrm{KhR}^{\Sigma}(K^k)$ corresponding to $\{\lambda_1^a, \lambda_2^b\} \subset \Sigma$.
Want to split it into tensor factors corresponding to $\{\lambda_1^a\}$, $\{\lambda_2^b\}$.



Proof Step 4 – The $\bigotimes$ decomposition

4. Look at summand of $\mathrm{KhR}^{\Sigma}(K^k)$ corresponding to $\{\lambda_1^a, \lambda_2^b\} \subset \Sigma$. Want to split it into tensor factors corresponding to roots λ_1, λ_2.



Proposition

- *This root-splitting process works for cube of resolutions chain complexes.*
- *They compute the same link invariants, but are manifestly tensor products of their root-colored parts.*

Proof Step 5 – Identifying the tensor factors

5. The tensor factors from the previous step are complexes in the subcategory $N\mathbf{Foam}^{\lambda_j\in\Sigma}$ of $N\mathbf{Foam}^{\Sigma}$, which consists of foams where every k-facet is decorated by the $\{\lambda_j^k\}$-idempotent.

Lemma

- *$N\mathbf{Foam}^{\lambda_j\in\Sigma}$ is isomorphic to $N_j\mathbf{Foam}^{\bullet}$.*
- *The isomorphism sends the λ_j tensor factor from the previous step to the cube of resolutions complex computing $\mathrm{KhR}^{N_j}(K^{k_j})$.*

This finishes the proof.

Plan

1. Motivation
2. sl(N) link homologies
3. Deformations
4. Physical structure, HOMFLY-PT homology

Large N limit

Physical expectation: $\mathfrak{sl}_N$ homologies have a large N limit. Problem!

- 2004: Khovanov-Rozansky: reduced Khovanov Rozansky homology categorifies the reduced $\mathfrak{sl}_N$ polynomial.

$$\widetilde{\mathrm{KhR}}^N(\bigcirc) \cong \mathbb{C}$$

- 2005: Khovanov-Rozansky: reduced triply-graded HOMFLY-PT homology categorifies the reduced HOMFLY-PT polynomial.

$$\widetilde{\mathrm{KhR}}^\infty(\bigcirc) \cong \mathbb{C}$$

- 2006: Rasmussen: for a knot K there exist spectral sequences

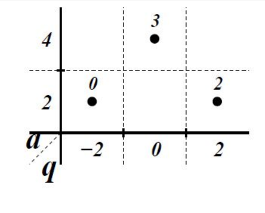
$$\widetilde{\mathrm{KhR}}^\infty(K)|_{a=q^N} \rightsquigarrow \widetilde{\mathrm{KhR}}^N(K)$$

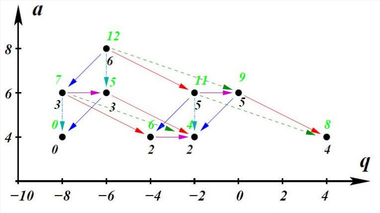
  which become trivial for large N.
- 2016: W.: add “colored” in the above.

More physical predictions

Large N stabilization is part of a system of expected relationships between reduced colored $\mathfrak{sl}_N$ and HOMFLY-PT homologies. Other main features:

- Differentials: $\widetilde{\mathrm{KhR}}^N(K) \rightsquigarrow \widetilde{\mathrm{KhR}}^M(K)$ for $N \geq M$
- Exponential growth: $\widetilde{\mathrm{KhR}}^\infty(K^k) \cong \left(\widetilde{\mathrm{KhR}}^\infty(K^1)\right)^{\otimes k}$
  for sufficiently simple knots K, after collapsing the q-grading
- Symmetries





Graphics from ↓

Dunfield-Gukov-Rasmussen 2005, Gukov-Stošić 2011, Gorsky-Gukov-Stošić 2013, Gukov-Nawata-Saberi-Stošić-Sułkowski 2016.

Deformations and differentials

Theorem (W. 2016)

Let K be a knot, $\sum N_j = N$ with $N_j \in \mathbb{N}$, $\sum k_j = k$ with $k_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a spectral sequence:

$$\widetilde{\mathrm{KhR}}^N(K^k) \rightsquigarrow \bigotimes_j \widetilde{\mathrm{KhR}}^{N_j}(K^{k_j})$$

Corollary (differentials)

Let K be a knot and $N \geq M$. There exists a spectral sequence:

$$\widetilde{\mathrm{KhR}}^N(K) \rightsquigarrow \widetilde{\mathrm{KhR}}^M(K)$$

Deformations and exponential growth

Theorem (W. 2016)

Let K be a knot, $\sum N_j = N$ with $N_j \in \mathbb{N}$, $\sum k_j = k$ with $k_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a spectral sequence:

$$\widetilde{\mathrm{KhR}}^N(K^k) \rightsquigarrow \bigotimes_j \widetilde{\mathrm{KhR}}^{N_j}(K^{k_j})$$

Corollary ($\geq$ exponential growth)

Let K be a knot and $k \in \mathbb{N}$. There exist spectral sequences:

$$\begin{array}{ccc} \widetilde{\mathrm{KhR}}^\infty(K^k) & & (\widetilde{\mathrm{KhR}}^\infty(K^1))^{\otimes k} \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{\mathrm{KhR}}^{kN}(K^k) & \rightsquigarrow & (\widetilde{\mathrm{KhR}}^N(K^1))^{\otimes k} \end{array} \quad \text{for } N \gg 0\,.$$

Further directions

- Deformations help to prove the functoriality of colored $\mathfrak{sl}_N$ homology under link cobordisms, following an idea of Blanchet.
- Deformed reduced link homologies produce interesting new slice genus bounds, Lewark-Lobb.
- What is q-holonomicity for link homologies?
- The remaining features of the conjectured physical structure motivate the development of $\mathfrak{gl}_{M|N}$ Lie superalgebra link homologies.
- Relations to link homologies of a more analytic flavor, e.g. Ozsváth-Szabó, Rasmussen knot Floer homology, which is a $\mathfrak{gl}_{1|1}$ link homology, Ellis-Petkova-Vértesi.
- Link homologies in other 3-manifolds, categorified Witten-Reshetikhin-Turaev invariants, 4d TQFTs...