Some differentials on colored Khovanov-Rozansky link homology

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Knots in Hellas 2016

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Plan

Reshetikhin-Turaev link invariants

The Reshetikhin-Turaev invariants for links in $\mathbb{R}^3$ give a function:

$$\{\text{triples } (L, \mathfrak{g}, \mathrm{col})\} \xrightarrow{\mathrm{RT}} \mathbb{C}(q)$$

- L is a framed, oriented link in $\mathbb{R}^3$,
- $\mathfrak{g}$ is a complex semi-simple Lie algebra,
- $\mathrm{col}\colon \pi_0(L) \to \mathrm{Irrep}^{f.d.}(\mathfrak{g})$ is a coloring of the link components by finite-dimensional irreducible representations of $\mathfrak{g}$.

Question

How does this function depend on the three arguments?

For this talk:

- Lie algebras are of type A: $\mathfrak{g} = \mathfrak{sl}_N$ for various $N \in \mathbb{N}$.
- Mostly colorings by irreps $\mathbb{C}^N$ and $\bigwedge^k \mathbb{C}^N$ for $0 \leq k \leq N$.

Varying the Lie algebra

Fact

The Jones polynomial (which appears as $\mathrm{RT}(-, \mathfrak{sl}_2, \mathbb{C}^2)$) is uniquely determined by its value on the unknot and the oriented skein relation:

$$q^2 V(\nwarrow\!\!\!\!\nearrow) - q^{-2} V(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) V(\uparrow\uparrow)$$

Varying this skein relation, we get other link polynomials:

- $q^N P_N(\nwarrow\!\!\!\!\nearrow) - q^{-N} P_N(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) P_N(\uparrow\uparrow)$ for $\mathrm{RT}(-, \mathfrak{sl}_N, \mathbb{C}^N)$.
- $a P_\infty(\nwarrow\!\!\!\!\nearrow) - a^{-1} P_\infty(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) P_\infty(\uparrow\uparrow)$
  for the HOMFLY-PT polynomial $\in \mathbb{Z}[a^{\pm 1}](q)$.
- $\Delta(\nwarrow\!\!\!\!\nearrow) - \Delta(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) \Delta(\uparrow\uparrow)$
  for the Alexander-Conway polynomial.

Varying the coloring

The finite-dimensional irreducible representations of $\mathfrak{sl}_2$ are indexed by $k \in \mathbb{N}$ (in fact $V_k := \mathrm{Sym}^k(\mathbb{C}^2)$). Redundancies in this countably-infinite list of invariants?

Theorem (Garoufalidis-Lê)

Let K be a framed knot in $\mathbb{R}^3$. The sequence of colored Jones polynomials $(\mathrm{RT}(K, \mathfrak{sl}_2, \mathrm{Sym}^k(\mathbb{C}^2)))_{k\in\mathbb{N}}$ is q-holonomic.

So the sequence is governed by a linear recurrence relation (with coefficients polynomials in q and q^k) and, thus, determined by a finite part.

Analogous results hold for $\mathfrak{sl}_N$, for colored HOMFLY-PT polynomials, for links, with other sequences of colors... Garoufalidis-Lauda-Lê.

Varying the link?

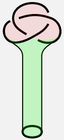
Lie algebras and colorings can be varied in families. Some links come in families too, but let's take a different perspective.
Instead of just links, consider link embeddings in $\mathbb{R}^3$ and smooth cobordisms between them (in $\mathbb{R}^3 \times I$). Need categorified RT invariants:

link homology LH ⇢ $\left\{ \begin{array}{c} \text{bi-graded vector spaces} \\ H_*(\text{graded chain complex}) \end{array} \right\}$

$$\{\mathfrak{g}\text{-colored oriented links}\} \xrightarrow{\mathrm{RT}} \mathbb{C}(q)$$

with χ_q: $\{\text{bi-graded vector spaces}\} \to \mathbb{C}(q)$

Cobordisms:



$LH(3_1)$

↑

$LH(0_1)$

Ideally functorial under link cobordisms.

Goal for this talk

Overview about the rank- and color-dependence of type A link homologies.

Plan

Khovanov homology and its cousins

- 1999: Khovanov homology categorifies the Jones polynomial.

$$\mathrm{Kh}(\bigcirc) \cong H^*(\mathbb{CP}^1)\{-1\}$$

- 2004: Khovanov-Rozansky homology categorifies $\mathrm{RT}(-, \mathfrak{sl}_N, \mathbb{C}^N)$.

$$\mathrm{KhR}^N(\bigcirc) \cong H^*(\mathbb{CP}^{N-1})\{1-N\}$$

- 2009: Wu and Yonezawa extended Khovanov-Rozansky homology to a categorification of $\mathrm{RT}(-, \mathfrak{sl}_N, \bigwedge^k \mathbb{C}^N)$.

$$\mathrm{KhR}^N(\bigcirc^k) \cong H^*(\mathrm{Gr}(k,N))\{k(k-N)\}$$

$$\mathrm{KhR}^N(K^1) = \mathrm{KhR}^N(K)$$

Flavors of colored $\mathfrak{sl}_N$ link homologies

1. Vanilla: via matrix factorizations, Khovanov-Rozansky, Wu, Yonezawa.
2. Representation theoretic: via category $\mathcal{O}$, Mazorchuk-Stroppel, Sussan.
3. Combinatorial: via cobordism or foam categories, Bar-Natan, Khovanov, Mackaay-Stošić-Vaz, Lauda-Queffelec-Rose.
4. Algebro-geometric: via affine Grassmannians, Cautis-Kamnitzer-Licata
5. Diagram-algebraic: via categorified tensor products, Webster.
6. Symplectic: via Floer homology, Seidel-Smith, Manolescu, Abouzaid.
7. Physical: via BPS state counting, Gukov-Schwarz-Vafa, et.al.

Two questions about the $\mathfrak{sl}_N$ link homology family

- What kind of geometric and topological information is accessible to it?
- What relations exist between its members?

Geometric and topological information

- Concordance homomorphisms, slice genus bounds, Rasmussen, Lobb, Wu.
- Thurston-Bennequin number bounds, Shumakovitch, Plamenevskaya, Ng.
- Splitting number bounds, Batson-Seed.
- Unknot detection, Kronheimer-Mrowka.
- ...

Fact

These results rely on spectral sequences between different link homologies.

Relations via deformation spectral sequences

- 2002: Lee constructed spectral sequences

$$\mathrm{Kh}(K) \rightsquigarrow \mathbb{C}^2 \qquad \mathrm{Kh}(L) \rightsquigarrow \mathbb{C}^{2^{|\pi_0(L)|}}$$

  leading to Rasmussen's concordance homomorphism.

- 2004: Gornik constructed spectral sequences

$$\mathrm{KhR}^N(K) \rightsquigarrow \mathbb{C}^N \qquad \mathrm{KhR}^N(L) \rightsquigarrow \mathbb{C}^{N^{|\pi_0(L)|}}$$

  leading to Lobb's concordance homomorphism.

- 2006: Mackaay-Vaz constructed spectral sequences:

$$\mathrm{KhR}^3(K) \rightsquigarrow \mathrm{KhR}^2(K) \oplus \mathbb{C}$$

More deformations

Theorem (folklore)

Let K be a knot and $\sum N_j = N$ with $N_j \in \mathbb{N}$, then there exists a deformation spectral sequence:

$$\mathrm{KhR}^N(K) \rightsquigarrow \bigoplus_j \mathrm{KhR}^{N_j}(K)$$

Theorem (Rose-W. 2015)

Let K be a knot and $\sum N_j = N$ with $N_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a deformation spectral sequence:

$$\mathrm{KhR}^N(K^k) \rightsquigarrow \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}^{N_j}(K^{k_j})$$

Mutatis mutandis for links.

Proof strategy

Theorem (Rose-W. 2015)

Let K be a knot and $\sum N_j = N$ with $N_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a deformation spectral sequence:

$$\mathrm{KhR}^N(K^k) \rightsquigarrow \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}^{N_j}(K^{k_j})$$

1. Wu's spectral sequence
2. Unknot case
3. $\bigoplus$ decomposition
4. $\bigotimes$ decomposition
5. Identifying tensor factors

Proof Step 1 – Wu's spectral sequence

1. Wu's construction of colored $\mathfrak{sl}_N$ homology uses matrix factorization with potential X^N.

   Following ideas of Gornik and Rasmussen:
   Potential $P(X) = \prod_{\lambda \in \Sigma}(X - \lambda) \in \mathbb{C}[X]$ of degree N with root multiset Σ gives a singly-graded, filtered link homology theory $\mathrm{KhR}^{\Sigma}(-)$ and spectral sequences

   $$\mathrm{KhR}^{N}(K^k) \rightsquigarrow \mathrm{KhR}^{\Sigma}(K^k)$$

   It remains to compute $\mathrm{KhR}^{\Sigma}(K^k)$ in terms of undeformed homologies.

Proof Step 2 – The unknot case

2. The link homology theory $\mathrm{KhR}^{\Sigma}(-)$ contains – and is controlled by – a $(1+1)$-dimensional TQFT. The corresponding commutative Frobenius algebra appears as the unknot invariant.
   Let $\Sigma = \{\lambda_1^{N_1}, \ldots, \lambda_l^{N_l}\}$, $P(X) = \prod_j (X - \lambda_j)^{N_j}$, then we have:

$$\mathrm{KhR}^{\Sigma}(\bigcirc^1) \cong \frac{\mathbb{C}[X]}{\langle P(X) \rangle} \cong \bigoplus_j \frac{\mathbb{C}[X]}{\langle (X-\lambda_j)^{N_j} \rangle} \cong \bigoplus_j \mathrm{KhR}^{N_j}(\bigcirc^1).$$

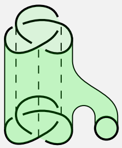
   Summands are indexed by roots of $P(X)$. And in the colored case:

$$\mathrm{KhR}^{\Sigma}(\bigcirc^k) \cong \frac{\mathrm{Sym}[\mathbb{X}]}{\langle h_{N-k+i}(\mathbb{X}-\Sigma) \mid i > 1 \rangle} \cong \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}^{N_j}(\bigcirc^{k_j}).$$

   Summands are indexed by size k multisubsets $\{\lambda_1^{k_1}, \ldots, \lambda_l^{k_l}\}$ of roots.

Proof Step 3 – The $\bigoplus$ decomposition

3. $\mathrm{KhR}^{\Sigma}(K^k)$ is a $\mathrm{KhR}^{\Sigma}(\bigcirc^k)$-module. If you believe in functoriality:



4. The proof of the $\bigotimes$ decomposition and
5. the identification of the tensor factors depend heavily on the particular link homology construction. Here: $\mathfrak{sl}_N$-foams of Queffelec-Rose coupled to Karoubi envelope techniques inspired by Bar-Natan Morrison.

Plan

Large N limit

Physical expectation: $\mathfrak{sl}_N$ homologies have a large N limit.

- 2004: Khovanov-Rozansky: reduced Khovanov Rozansky homology categorifies the reduced $\mathfrak{sl}_N$ polynomial.

$$\widetilde{\mathrm{KhR}}^N(\bigcirc) \cong \mathbb{C}$$

- 2005: Khovanov-Rozansky: reduced HOMFLY-PT homology categorifies the reduced HOMFLY-PT polynomial.

$$\widetilde{\mathrm{KhR}}^\infty(\bigcirc) \cong \mathbb{C}$$

- 2006: Rasmussen: for a knot K there exist spectral sequences

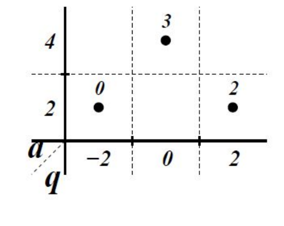
$$\widetilde{\mathrm{KhR}}^\infty(K)|_{a=q^N} \rightsquigarrow \widetilde{\mathrm{KhR}}^N(K)$$

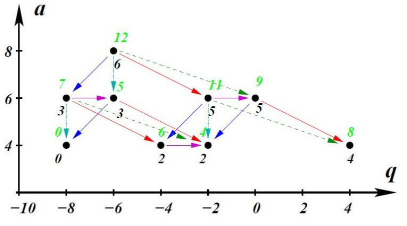
  which become trivial for large N.
- 2016: W.: add “colored” in the above.

More physical predictions

Large N stabilization is part of a system of expected relationships between reduced colored $\mathfrak{sl}_N$ and HOMFLY-PT homologies. Other main features:

- Differentials: $\widetilde{\mathrm{KhR}}^N(K) \rightsquigarrow \widetilde{\mathrm{KhR}}^M(K)$ for $N \geq M$
- Exponential growth: $\widetilde{\mathrm{KhR}}^\infty(K^k) \cong \left(\widetilde{\mathrm{KhR}}^\infty(K^1)\right)^{\otimes k}$ for sufficiently simple knots K, after collapsing the q-grading
- Symmetries





Dunfield-Gukov-Rasmussen 2005, Gukov-Stošić 2011, Gorsky-Gukov-Stošić 2013, Gukov-Nawata-Saberi-Stošić-Sułkowski 2016.

Deformations and differentials

Theorem (W. 2016)

Let K be a knot, $\sum N_j = N$ with $N_j \in \mathbb{N}$, $\sum k_j = k$ with $k_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a spectral sequence:

$$\widetilde{\mathrm{KhR}}^N(K^k) \rightsquigarrow \bigotimes_j \widetilde{\mathrm{KhR}}^{N_j}(K^{k_j})$$

Corollary (differentials)

Let K be a knot and $N \geq M$. There exists a spectral sequence:

$$\widetilde{\mathrm{KhR}}^N(K) \rightsquigarrow \widetilde{\mathrm{KhR}}^M(K)$$

Deformations and exponential growth

Theorem (W. 2016)

Let K be a knot, $\sum N_j = N$ with $N_j \in \mathbb{N}$, $\sum k_j = k$ with $k_j \in \mathbb{N}$, and write K^k for K colored by $\bigwedge^k \mathbb{C}^N$, then there exists a spectral sequence:

$$\widetilde{\mathrm{KhR}}^N(K^k) \rightsquigarrow \bigotimes_j \widetilde{\mathrm{KhR}}^{N_j}(K^{k_j})$$

Corollary ($\geq$ exponential growth)

Let K be a knot and $k \in \mathbb{N}$. There exist spectral sequences:

$$\begin{array}{ccc} \widetilde{\mathrm{KhR}}^\infty(K^k) & & (\widetilde{\mathrm{KhR}}^\infty(K^1))^{\otimes k} \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{\mathrm{KhR}}^{kN}(K^k) & \rightsquigarrow & (\widetilde{\mathrm{KhR}}^N(K^1))^{\otimes k} \end{array} \quad \text{for } N \gg 0 .$$